

THE MONOTONICITY AND CONVEXITY OF A FUNCTION INVOLVING DIGAMMA ONE AND THEIR APPLICATIONS

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ABSTRACT. Let $\mathcal{L}(x, a)$ be defined on $(-1, \infty) \times (4/15, \infty)$ or $(0, \infty) \times (1/15, \infty)$ by the formula

$$\mathcal{L}(x, a) = \frac{1}{90a^2+2} \ln \left(x^2 + x + \frac{3a+1}{3} \right) + \frac{45a^2}{90a^2+2} \ln \left(x^2 + x + \frac{15a-1}{45a} \right).$$

We investigate the monotonicity and convexity of the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$, where ψ denotes the Psi function. And, we determine the best parameter a such that the inequality $\psi(x+1) < (>) \mathcal{L}(x, a)$ holds for $x \in (-1, \infty)$ or $(0, \infty)$, and then, some new and very high accurate sharp bounds for psi function and harmonic numbers are presented. As applications, we construct a sequence $(l_n(a))$ defined by $l_n(a) = H_n - \mathcal{L}(n, a)$, which gives extremely accurate values for γ .

1. INTRODUCTION

For $x > 0$ the classical Euler's gamma function Γ and psi (digamma) function ψ are defined by

$$(1.1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The derivatives $\psi', \psi'', \psi''', \dots$ are known as polygamma functions (see [10]).

The gamma and polygamma functions play a central role in the theory of special functions and have extensive applications in many branches, such as mathematical physics, probability, statistics, engineering. In the recent past, numerous papers have appeared providing various inequalities for gamma and polygamma functions. A detailed list of references is given in [26]. In addition, some new results can be found in [6], [7], [8], [9], [11], [12], [13], [14], [15], [16], [19], [17], [18], [20], [23], [25], [27], [34], [36], [37], [39], and closely-related references therein.

In particular, we mention the following inequalities proved by Batir [14, Lemma 1.7]

$$(1.2) \quad \ln \left(x + \frac{1}{2} \right) < \psi(x+1) \leq \ln \left(x + e^{-\gamma} \right)$$

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for $x > 0$, where $1/2$ and $e^{-\gamma}$ are the best possible constants, $\gamma = 0.577215664 \dots$ is the Euler-Mascheroni constant. Let H_n denote the harmonic number defined by

$$(1.3) \quad H_n = \sum_{k=1}^n \frac{1}{k}, \quad (n \in \mathbb{N}).$$

From (1.2) and the relation $H_n = \gamma + \psi(n+1)$ (see [1, p.258]) it follows that

$$(1.4) \quad \gamma + \ln\left(n + \frac{1}{2}\right) < H_n \leq \gamma + \ln\left(n + e^{1-\gamma} - 1\right)$$

hold for $n \in \mathbb{N}$. In 2011, He showed further in [15, Corollary 2.2] that

$$(1.5) \quad \frac{1}{2} \ln(x^2 + x + e^{-2\gamma}) \leq \psi(x+1) < \frac{1}{2} \ln(x^2 + x + \frac{1}{3})$$

is valid for all $x > 0$, where $e^{-2\gamma}$ and $1/3$ are the best possible. As a direct consequence, he obtained for $n \in \mathbb{N}$.

$$(1.6) \quad \gamma + \frac{1}{2} \ln(n^2 + n + e^{2-2\gamma} - 2) \leq H_n < \gamma + \frac{1}{2} \ln(n^2 + n + \frac{1}{3}).$$

Furthermore, for $n \in \mathbb{N}$, the sequence (σ_n) defined by

$$(1.7) \quad \sigma_n = H_n - \frac{1}{2} \ln(n^2 + n + \frac{1}{3})$$

is strictly increasing and

$$\lim_{n \rightarrow \infty} n^4 (\sigma_n - \gamma) = -\frac{1}{180},$$

which means that the sequence (σ_n) converges to γ like n^{-4} .

Motivated by the above results, we easily construct a real function $(x, a) \rightarrow \mathcal{L}(x, a)$ defined on $(-1, \infty) \times (4/15, \infty)$ or $(0, \infty) \times (1/15, \infty)$ by the formula

$$(1.8) \quad \mathcal{L}(x, a) = \frac{1}{90a^2+2} \ln(x^2 + x + \frac{3a+1}{3}) + \frac{45a^2}{90a^2+2} \ln(x^2 + x + \frac{15a-1}{45a}).$$

By arithmetic-geometric mean inequality we have

$$\begin{aligned} \mathcal{L}(x, a) &= \frac{1}{2} \ln \left(\left(x^2 + x + \frac{3a+1}{3} \right)^{1/(45a^2+1)} \left(x^2 + x + \frac{15a-1}{45a} \right)^{45a^2/(45a^2+1)} \right) \\ &\leq \frac{1}{2} \ln \left(\frac{1}{45a^2+1} \left(x^2 + x + \frac{3a+1}{3} \right) + \frac{45a^2}{45a^2+1} \left(x^2 + x + \frac{15a-1}{45a} \right) \right) \\ &= \frac{1}{2} \ln \left(x^2 + x + \frac{1}{3} \right). \end{aligned}$$

From this, if the inequality $\psi(x+1) \leq \mathcal{L}(x, a)$ is valid for $x \in (-1, \infty)$, then the second inequality of (1.5) may be refined.

It is clear that

$$(1.9) \quad \lim_{x \rightarrow \infty} (\psi(x+1) - \mathcal{L}(x, a)) = 0,$$

and employing L'Hospital's rule together with

$$(1.10) \quad \psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \dots \quad (\text{as } x \rightarrow \infty)$$

(see [1, pp. 258–260.]), we easily get the following limit relations:

$$(1.11) \quad \lim_{x \rightarrow \infty} \frac{\psi(x+1) - \mathcal{L}(x, a)}{x^{-6}} = -\frac{\left(a - \frac{40+3\sqrt{205}}{105}\right)\left(a - \frac{40-3\sqrt{205}}{105}\right)}{85050a},$$

$$(1.12) \quad \lim_{x \rightarrow \infty} \frac{\psi(x+1) - \mathcal{L}(x, a_1)}{x^{-8}} = -\frac{2}{1225},$$

where $a_1 = (40 + 3\sqrt{205})/105$. (1.11) and (1.12) indicates that $\psi(x+1) - \mathcal{L}(x, a)$ and $\psi(x+1) - \mathcal{L}(x, a_1)$ converge to zero as x tends infinite like x^{-6} and x^{-8} , respectively.

The first aim of this paper is to determine the best a such that the function $F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ has monotonicity and convexity properties, which are showed in Section 3. The second aim is to determine the best a such that the inequality

$$(1.13) \quad \psi(x+1) < \mathcal{L}(x, a)$$

holds for $x \in (-1, \infty)$ and its reverse holds for $x \in (0, \infty)$, which yield some new sharp bounds for harmonic numbers H_n , and they are presented in Section 4. In Section 5, as an application, we construct a sequence $(l_n(a))$ defined by $l_n(a) = H_n - \mathcal{L}(n, a)$, which gives extremely accurate values for γ and greatly improves some known results. Lastly, an open problem is posted.

Some complicated algebraic computations are preformed with the aid of built-in computer algebra system of *Scientific Workplace Version 5.5*.

2. LEMMAS

Lemma 1. *Let $\mathcal{L}(x, a)$ be defined by the formula (1.8).*

(i) *If $x > -1$ then the function \mathcal{L} is increasing in a on $(4/15, \infty)$, and*

$$(2.1) \quad \lim_{a \rightarrow \infty} \mathcal{L}(x, a) = \frac{1}{2} \ln(x^2 + x + \frac{1}{3}).$$

(ii) *If $x > 0$ then $a \mapsto \partial \mathcal{L} / \partial x$ is decreasing, $a \mapsto \partial^2 \mathcal{L} / \partial x^2$ is increasing and $a \mapsto \partial^3 \mathcal{L} / \partial x^3$ is decreasing on $(1/15, \infty)$.*

Proof. (i) Direct partial derivative calculations yield

$$(2.2) \quad \begin{aligned} \frac{\partial \mathcal{L}}{\partial a} &= \frac{45a}{(45a^2+1)^2} \ln \frac{x^2+x+(15a-1)/(45a)}{x^2+x+(3a+1)/3} \\ &\quad + \frac{1}{(90a^2+2)} \frac{1}{x^2+x+(3a+1)/3} + \frac{1}{90a^2+2} \frac{1}{x^2+x+(15a-1)/(45a)}, \\ \frac{\partial^2 \mathcal{L}}{\partial x \partial a} &= -\frac{45a^2+1}{4050a^2} \frac{2x+1}{(x^2+x+(3a+1)/3)^2 (x^2+x+(15a-1)/(45a))^2}, \end{aligned}$$

Clearly, if $x \geq -1/2$, then $\partial^2 \mathcal{L} / \partial x \partial a < 0$, which means that $\partial \mathcal{L} / \partial a$ decreases with x . This leads to

$$\frac{\partial \mathcal{L}(x, a)}{\partial a} > \lim_{x \rightarrow \infty} \frac{\partial \mathcal{L}(x, a)}{\partial a} = 0,$$

that is, $\mathcal{L}(x, a)$ increases with a on $(4/15, \infty)$.

If $-1 < x < -1/2$, then $\partial^2 \mathcal{L} / \partial x \partial a > 0$, that is, $\partial \mathcal{L} / \partial a$ increases with x on $(-1, -1/2)$. Hence,

$$\begin{aligned} \frac{\partial \mathcal{L}(x, a)}{\partial a} &> \lim_{x \rightarrow -1+} \frac{\partial \mathcal{L}(x, a)}{\partial a} \\ &= \frac{45a}{(45a^2+1)^2} \left(\frac{45a^2+1}{90a} \frac{135a^2+90a-3}{45a^2+12a-1} + \ln \left(\frac{(15a-1)}{15a(3a+1)} \right) \right) \\ &:= \frac{45a}{(45a^2+1)^2} \mathcal{L}_1(a). \end{aligned}$$

An elementary computation gives

$$\mathcal{L}'_1(a) = \frac{(45a^2+1)^2}{30a^2(45a^2+12a-1)^2} \left(a + \frac{\sqrt{6}-1}{15} \right) \left(a - \frac{\sqrt{6}+1}{15} \right) > 0$$

for $a \in (4/15, \infty)$, which indicates that $\mathcal{L}_1(a)$ increases with a . Consequently,

$$\begin{aligned}\mathcal{L}_1(a) &> \mathcal{L}_1\left(\frac{4}{15}\right) = \left[\frac{45a^2+1}{90a} \frac{135a^2+90a-3}{45a^2+12a-1} + \ln\left(\frac{(15a-1)}{15a(3a+1)}\right) \right]_{a=4/15} \\ &= \ln \frac{5}{12} + \frac{119}{120} > 0,\end{aligned}$$

it follows that

$$\frac{\partial \mathcal{L}}{\partial a}(x, a) > \frac{45a}{(45a^2+1)^2} \mathcal{L}_1(a) > 0.$$

Thus, we have $\partial \mathcal{L} / \partial a > 0$ for $x \in (-1, \infty)$ and $a \in (4/15, \infty)$.

(ii) By (2.2) it is clear that $\partial^2 \mathcal{L} / \partial x \partial a = \partial^2 \mathcal{L} / \partial a \partial x < 0$ if $x > 0$, and so $\partial \mathcal{L} / \partial x$ decreases with a .

Partial derivative computations once again give

$$(2.3) \quad \mathcal{L}_x = \frac{\partial \mathcal{L}}{\partial x} = \frac{1}{90a^2+2} \frac{2x+1}{x^2+x+a+1/3} + \frac{45a^2}{90a^2+2} \frac{2x+1}{x^2+x+(15a-1)/(45a)},$$

$$(2.4) \quad \mathcal{L}_{xx} = \frac{\partial^2 \mathcal{L}}{\partial x^2} = -\frac{1}{45a^2+1} \frac{x^2+x-a+1/6}{(x^2+x+a+1/3)^2} - \frac{45a^2}{45a^2+1} \frac{x^2+x+1/(45a)+1/6}{(x^2+x+(15a-1)/(45a))^2},$$

$$\frac{\partial^3 \mathcal{L}}{\partial x^2 \partial a} = \frac{45a^2+1}{3^4 5^3 a^3} \frac{P(x)}{(x^2+x+(3a+1)/3)^3 (x^2+x+(15a-1)/(45a))^3},$$

$$(2.5) \quad \mathcal{L}_{xxx} = \frac{\partial^3 \mathcal{L}}{\partial x^3} = \frac{1}{45a^2+1} \frac{(2x+1)(x^2+x-3a)}{(x^2+x+a+1/3)^3} + \frac{45a^2}{45a^2+1} \frac{(2x+1)(x^2+x+1/(15a))}{(x^2+x+(15a-1)/(45a))^3},$$

$$\frac{\partial^4 \mathcal{L}}{\partial x^3 \partial a} = -\frac{45a^2+1}{3^8 5^4} \frac{(2x+1) \times Q(x)}{(x^2+x+a+1/3)^4 (x^2+x+(15a-1)/(45a))^4},$$

where

$$\begin{aligned}P(x) &= 945ax^4 + 1890ax^3 + (405a^2 + 1485a - 9)x^2 \\ &\quad + (405a^2 + 540a - 9)x + (90a^2 + 78a - 2),\end{aligned}$$

$$Q(x) = (56\,700a^2)x^6 + (170\,100a^2)x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

here

$$\begin{aligned}b_4 &= (44\,550a^3 + 220\,050a^2 - 990a), \\ b_3 &= 89\,100a^3 + 156\,600a^2 - 1980a, \\ b_2 &= (12\,150a^4 + 67\,500a^3 + 64\,710a^2 - 1500a + 6), \\ b_1 &= (12\,150a^4 + 22\,950a^3 + 14\,760a^2 - 510a + 6), \\ b_0 &= (2025a^4 + 2970a^3 + 1530a^2 - 66a + 1).\end{aligned}$$

It is easy to verify that all coefficients of $P(x)$ and $Q(x)$ are positive for $a \in (1/15, \infty)$, which leads to $\partial^3 \mathcal{L} / \partial x^2 \partial a > 0$, $\partial^4 \mathcal{L} / \partial x^3 \partial a < 0$ for $x > 0$, which proves the desired results. \square

Remark 1. By the second assertion in the above lemma, and making use of mean-value theorem, we see that the following functions

$$(2.6) \quad \begin{aligned}a &\mapsto \mathcal{L}(x, a) - \mathcal{L}(y, a), \\ a &\mapsto \mathcal{L}_x(x, a) - \mathcal{L}_x(y, a), \\ a &\mapsto \mathcal{L}_{xx}(x, a) - \mathcal{L}_{xx}(y, a)\end{aligned}$$

are decreasing, increasing and decreasing on $(1/15, \infty)$ for $x > y > 0$.

Lemma 2 ([1, pp. 258–260.]). Let $x > 0$ and $n \in \mathbb{N}$. Then

$$(2.7) \quad \psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}},$$

The following lemma was first used to establish some monotonicity results for the gamma function [23], which also play an important role in proofs our main results.

Lemma 3 ([23]). *Let f be a function defined on an interval I and $\lim_{x \rightarrow \infty} f(x) = 0$. If $f(x+1) - f(x) > 0$ for all $x \in I$, then $f(x) < 0$. If $f(x+1) - f(x) < 0$ for all $x \in I$, then $f(x) > 0$.*

3. MONOTONICITY AND CONVEXITY

Theorem 1. *Let the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ be defined on $(-1, \infty)$ where $\mathcal{L}(x, a)$ be defined by 1.8. Then for $x > -1$, we have*

$$(-1)^n F_{a_1}^{(n)}(x) < 0, \quad n = 1, 2, 3,$$

where $a_1 = (40 + 3\sqrt{205})/105 \approx 0.79003$.

Proof. Differentiation yields

$$(3.1) \quad F'_a(x) = \psi'(x+1) - \mathcal{L}_x(x, a),$$

$$(3.2) \quad F''_a(x) = \psi''(x+1) - \mathcal{L}_{xx}(x, a),$$

$$(3.3) \quad F'''_a(x) = \psi'''(x+1) - \mathcal{L}_{xxx}(x, a),$$

where $\mathcal{L}_x(x, a)$, $\mathcal{L}_{xx}(x, a)$ and $\mathcal{L}_{xxx}(x, a)$ are given by (2.3), (2.4) and (2.5), respectively

Clearly, we have

$$(3.4) \quad \lim_{x \rightarrow \infty} F'_a(x) = \lim_{x \rightarrow \infty} F''_a(x) = \lim_{x \rightarrow \infty} F'''_a(x) = 0.$$

From the relation (2.7), it is deduced that

$$\begin{aligned} F'_a(x+1) - F'_a(x) &= \psi'(x+2) - \psi'(x+1) - \mathcal{L}_x(x+1, a) + \mathcal{L}_x(x, a) \\ &= -\frac{1}{(x+1)^2} - \frac{2(x+1)+1}{(90a^2+2)((x+1)^2+(x+1)+(3a+1)/3)} - \frac{45a^2(2(x+1)+1)}{(90a^2+2)((x+1)^2+(x+1)+(15a-1)/(45a))} \\ &\quad + \frac{2x+1}{(90a^2+2)(x^2+x+(3a+1)/3)} + \frac{45a^2(2x+1)}{(90a^2+2)(x^2+x+(15a-1)/(45a))}, \end{aligned}$$

which, by factoring and simplifying, can be written as

$$(3.5) \quad F'_a(x+1) - F'_a(x) = \frac{q(x, a)}{p(x, a)},$$

where

$$(3.6) \quad q(x, a) = \frac{315 \left(a + \frac{3\sqrt{205}-40}{105} \right) (a_1 - a)}{2025a} (x+1)^2 - \frac{(a+1/3)^2 (a-1/15)^2}{9a^2}$$

$$(3.7) \quad p(x, a) = (x+1)^2 (x^2 + 3x + (a+7/3)) (x^2 + x + (a+1/3)) \cdot (x^2 + x + 1/3 - 1/(45a)) (x^2 + 3x + 7/3 - 1/(45a)) > 0$$

Substituting $a = a_1 = (40 + 3\sqrt{205})/105$ into (3.6) yields

$$q(x, a_1) = -\frac{144}{1225},$$

and then

$$F'_{a_1}(x+1) - F'_{a_1}(x) = -\frac{144}{1225} \frac{1}{p(x, a_1)}.$$

Differentiation again yields

$$\begin{aligned}
F_{a_1}''(x+1) - F_{a_1}''(x) &= (F_{a_1}'(x+1) - F_{a_1}'(x))' = \frac{144}{1225} \frac{p_x(x, a_1)}{p^2(x, a_1)}, \\
F_{a_1}'''(x+1) - F_{a_1}'''(x) &= \frac{144}{1225} \frac{p_{xx}(x, a_1)p(x, a_1) - 2p_x^2(x, a_1)}{p^3(x, a_1)} \\
&= -\frac{144}{1225} \frac{2}{1500625} \frac{(x+1)^2}{p^3(x, a_1)} r(x),
\end{aligned}$$

where

$$\begin{aligned}
w(x) &= 82534375(x+1)^{16} + 111903750(x+1)^{14} + 117967500(x+1)^{12} \\
&\quad + 42925750(x+1)^{10} + 8270325(x+1)^8 - 12773700(x+1)^6 \\
&\quad + 3342880(x+1)^4 - 596160(x+1)^2 + 62208.
\end{aligned}$$

Since $p(x, a_1)$ is clearly positive, if we can prove $w(x) > 0$ for all $x > -1$, then we have $F_{a_1}'''(x+1) - F_{a_1}'''(x) < 0$, that is, $F_{a_1}'''(x) > F_{a_1}'''(x+1)$, which, by Lemma 3 together with $\lim_{x \rightarrow \infty} F_{a_1}'''(x) = 0$, yields

$$F_{a_1}'''(x) > \lim_{x \rightarrow \infty} F_{a_1}'''(x) = 0,$$

that is, $x \mapsto F_{a_1}''(x)$ is strictly increasing on $(-1, \infty)$, which results in

$$F_{a_1}''(x) < \lim_{x \rightarrow \infty} F_{a_1}''(x) = 0.$$

It is concluded that $x \mapsto F_{a_1}'(x)$ is strictly decreasing, and then

$$F_{a_1}'(x) > \lim_{x \rightarrow \infty} F_{a_1}'(x) = 0.$$

Hence, in order to deduce desired results, we have to show that $w(x) > 0$ for all $x > -1$. it is enough to prove $w_8(t) > 0$ for $t > 0$, where

$$\begin{aligned}
w_8(t) &= w(\sqrt{t} - 1) = 82534375t^8 + 111903750t^7 + 117967500t^6 + 42925750t^5 \\
&\quad + 8270325t^4 - 12773700t^3 + 3342880t^2 - 596160t + 62208.
\end{aligned}$$

Firstly, $w_8(t) > 0$ for $t \geq 1/8$. In fact, after replacing t by $(t_1 + 1/8)$ and expanding, we get

$$\begin{aligned}
w_8(t) &= (82534375t_1^8 + 194438125t_1^7 + \frac{4031873125}{16}t_1^6 + \frac{11337407375}{64}t_1^5 \\
&\quad + \frac{147062213725}{2048}t_1^4 + \frac{15458202275}{4096}t_1^3 + w_2(t_1), \\
w_2(t_1) &= \frac{44500267405}{65536}t_1^2 - \frac{56961842735}{262144}t_1 + \frac{315567169303}{16777216}.
\end{aligned}$$

Clearly, $w_8(t) - w_2(t_1) \geq 0$ due to $t_1 = t - 1/8 \geq 0$. While $w_2(t_1)$ is a quadratic polynomial, and by an ease check, the discriminant of the quadratic equation is negative and the coefficient of cubic term is positive, and therefore $w_2(t_1) > 0$. Thus we have $w_8(t) > 0$ for $t \geq 1/8$.

Secondly, we show that $w_8(t) > 0$ for $0 < t < 1/8$. Since the first five terms of eight degrees polynomial $w_8(t)$ is clearly positive, it suffices to prove that the last four terms of $w_8(t)$, that is, a cubic polynomial

$$w_3(t) := -12773700t^3 + 3342880t^2 - 596160t + 62208 > 0.$$

As $0 < t < 1/8$ we have

$$\begin{aligned} w_3(t) &> -12773700 \left(\frac{1}{8}\right)^3 + 3342880t^2 - 596160t + 62208 \\ &= 3342880t^2 - 596160t + \frac{4769199}{128} > 0, \end{aligned}$$

where the last inequality holds due to the discriminant of the quadratic equation is negative and the coefficient of quadratic term is positive.

This completes the proof. \square

Remark 2. From the proof of Theorem 1, we see that for $a \in (1/15, \infty)$

$$(3.8) \quad \frac{\partial q}{\partial a} = -\frac{7(45a^2+1)}{2025a^2} (x+1)^2 - \frac{2(a+1/3)(a-1/15)(2025a^2+45)}{18225a^3} < 0,$$

which shows that function $a \rightarrow q(x, a)$ is decreasing on $(1/15, \infty)$.

Remark 3. In the proof of Theorem 1, replacing x by $x - 1/2$ in (3.1)–(3.3) and simplifying yield

$$\begin{aligned} F'_{a_1}(x - \tfrac{1}{2}) &= \psi'(x + \tfrac{1}{2}) - \frac{20x(84x^2+71)}{1680x^4+1560x^2+81}, \\ F''_{a_1}(x - \tfrac{1}{2}) &= \psi''(x + \tfrac{1}{2}) + \frac{20}{3} \frac{47040x^6+75600x^4+30116x^2-1917}{(560x^4+520x^2+27)^2}, \\ F'''_{a_1}(x - \tfrac{1}{2}) &= \psi'''(x + \tfrac{1}{2}) - \frac{160}{3} \frac{x(6585600x^8+15052800x^6+11696160x^4+1820960x^2-701703)}{(560x^4+520x^2+27)^3} \end{aligned}$$

and utilization of Theorem 1, it is acquired directly that for $x > -1/2$, the inequalities

$$(3.9) \quad \psi'(x + \tfrac{1}{2}) > \frac{20x(84x^2+71)}{1680x^4+1560x^2+81},$$

$$(3.10) \quad \psi''(x + \tfrac{1}{2}) < -\frac{20}{3} \frac{47040x^6+75600x^4+30116x^2-1917}{(560x^4+520x^2+27)^2},$$

$$(3.11) \quad \psi'''(x + \tfrac{1}{2}) > \frac{160}{3} \frac{x(6585600x^8+15052800x^6+11696160x^4+1820960x^2-701703)}{(560x^4+520x^2+27)^3}$$

hold true.

Using Theorem 1 with Lemma 1, the following assertion is immediate.

Corollary 1. Let the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ be defined on $(0, \infty)$ where $\mathcal{L}(x, a)$ be defined by 1.8. Then for $x > 0$, F_a is increasing and concave if and only if $a \geq a_1 = (40 + 3\sqrt{205})/105 \approx 0.79003$.

Proof. The necessity follows from $\lim_{x \rightarrow \infty} x^7 F'_a(x) \geq 0$ and $\lim_{x \rightarrow \infty} x^8 F''_a(x) \leq 0$. Using L'Hospital's rule two times to the relation (1.11) give

$$\begin{aligned} -\frac{\left(a - \frac{40+3\sqrt{205}}{105}\right)\left(a - \frac{40-3\sqrt{205}}{105}\right)}{85050a} &= \lim_{x \rightarrow \infty} \frac{\psi'(x+1) - \mathcal{L}_x(x, a)}{-6x^{-7}} \leq 0, \\ -\frac{\left(a - \frac{40+3\sqrt{205}}{105}\right)\left(a - \frac{40-3\sqrt{205}}{105}\right)}{85050a} &= \lim_{x \rightarrow \infty} \frac{\psi''(x+1) - \mathcal{L}_{xx}(x, a)}{(-6)(-7)x^{-8}} \leq 0, \end{aligned}$$

which yield $a \geq a_1$.

By Theorem 1 with Lemma 1, we obtain that for $a \geq a_1$,

$$\begin{aligned} F'_a(x) &= \psi'(x+1) - \mathcal{L}_x(x, a) \geq \psi'(x+1) - \mathcal{L}_x(x, a_1) > 0, \\ F''_a(x) &= \psi''(x+1) - \mathcal{L}_{xx}(x, a) \leq \psi''(x+1) - \mathcal{L}_{xx}(x, a_1) < 0, \end{aligned}$$

which proves the sufficiency. \square

Theorem 2. Let the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ be defined on $(0, \infty)$ in which $\mathcal{L}(x, a)$ be defined by 1.8. Then F_a is decreasing on $(0, \infty)$ if and only if $a \in (1/15, a'_0]$, where

$$(3.12) \quad a'_0 = \frac{45 - 4\pi^2 + 3\sqrt{4\pi^4 - 80\pi^2 + 405}}{30(\pi^2 - 9)} \approx 0.47053.$$

Proof. Necessity. The necessity is deduced from

$$\begin{aligned} F'_a(0) &= \psi'(1) - \mathcal{L}_x(0, a) \\ &= \frac{\pi^2}{6} - \frac{1}{(90a^2 + 2)(a + 1/3)} - \frac{45a^2}{(90a^2 + 2)(15a - 1)/(45a)} \\ &= \frac{1}{6} \frac{45(\pi^2 - 9)a^2 - 3(45 - 4\pi^2)a - (\pi^2 - 9)}{(3a + 1)(15a - 1)} \leq 0, \end{aligned}$$

which in combination with $a \in (1/15, \infty)$ gives $a \in (1/15, a'_0]$.

Sufficiency. Due to Lemma 1, $a \mapsto \mathcal{L}_x(x, a)$ is decreasing on $(1/15, \infty)$, to prove the sufficiency, it is enough to prove $F'_{a'_0}(x) < 0$ for $x \in (0, \infty)$. We distinguish two cases to prove it.

Case 1: $x \in (1/20, \infty)$. From (3.6) and (3.8) and $a'_0 < 48/100$, we have

$$\begin{aligned} q(x, a'_0) &> q(x, \frac{48}{100}) > q(\frac{1}{20}, \frac{48}{100}) \\ &= \left[\frac{315(a + \frac{3\sqrt{205}-40}{105})(\frac{3\sqrt{205}+40}{105}-a)}{2025a} \left(\frac{1}{20} + 1 \right)^2 - \frac{(a+1/3)^2(a-1/15)^2}{9a^2} \right]_{a=48/100} \\ &= \frac{2341501}{1312200000} > 0, \end{aligned}$$

which in conjunction with (3.5) and (3.7) yields

$$F'_{a'_0}(x+1) - F'_{a'_0}(x) > 0.$$

Hence, we conclude that $F'_{a'_0}(x) < \lim_{x \rightarrow \infty} F'_{a'_0}(x) = 0$.

Case 2: $x \in (0, 1/20]$. If we show that $F''_{a'_0}(x) < 0$ for $x \in (0, 1/20]$, then we get $F'_{a'_0}(x) \leq F'_{a'_0}(0) = 0$, which proves the desired result. Replacing x by $x + 3/2$ in (3.10) and using (2.7), we have

$$\psi''(x+1) < -\frac{20}{3} \frac{47040(x+3/2)^6 + 75600(x+3/2)^4 + 30116(x+3/2)^2 - 1917}{(560(x+3/2)^4 + 520(x+3/2)^2 + 27)^2} - \frac{2}{(x+1)^3}.$$

Since $a \mapsto \mathcal{L}_{xx}(x, a_0)$ is increasing by Lemma 1 and $a'_0 \approx 0.47053 > 9/20$, we get

$$\mathcal{L}_{xx}(x, a'_0) > \mathcal{L}_{xx}(x, \frac{9}{20}) = -\frac{80}{809} \frac{x^2 + x - \frac{17}{60}}{(x^2 + x + \frac{47}{60})^2} - \frac{729}{809} \frac{x^2 + x + \frac{35}{162}}{(x^2 + x + \frac{23}{81})^2},$$

where $\mathcal{L}_{xx}(x, a)$ is given by (2.4). Thus, we have

$$\begin{aligned} F''_{a'_0}(x) &= \psi''(x+1) - \mathcal{L}_{xx}(x, a'_0) \\ &< -\frac{20}{3} \frac{47040(x+1+1/2)^6 + 75600(x+1+1/2)^4 + 30116(x+1+1/2)^2 - 1917}{(560(x+1+1/2)^4 + 520(x+1+1/2)^2 + 27)^2} \\ &\quad - \frac{2}{(x+1)^3} - \mathcal{L}_{xx}(x, \frac{9}{20}) \\ &= -\frac{20}{3} \frac{47040(x+1+1/2)^6 + 75600(x+1+1/2)^4 + 30116(x+1+1/2)^2 - 1917}{(560(x+1+1/2)^4 + 520(x+1+1/2)^2 + 27)^2} - \frac{2}{(x+1)^3} \\ &\quad + \frac{80}{809} \frac{x^2 + x - \frac{17}{60}}{(x^2 + x + \frac{47}{60})^2} + \frac{729}{809} \frac{x^2 + x + \frac{35}{162}}{(x^2 + x + \frac{23}{81})^2}. \end{aligned}$$

Factoring and arranging lead to

$$F''_{a'_0}(x) < \frac{1}{6} \frac{P(x)}{Q(x)},$$

where

$$\begin{aligned} P(x) = & 9756\,595\,800x^{11} + 146\,348\,937\,000x^{10} + 1005\,597\,383\,250x^9 \\ & + 3954\,619\,691\,700x^8 + 9800\,346\,642\,855x^7 + 16\,058\,808\,560\,085x^6 \\ & + 17\,731\,092\,059\,926x^5 + 13\,107\,900\,251\,862x^4 + 6210\,045\,031\,977x^3 \\ & + 1655\,666\,210\,995x^2 + 153\,061\,816\,584x - 15\,463\,394\,658, \end{aligned}$$

$$\begin{aligned} Q(x) = & (60x^2 + 60x + 47)^2 (81x^2 + 81x + 23)^2 \\ & \times (570x + 505x^2 + 210x^3 + 35x^4 + 252)^2 (x+1)^3. \end{aligned}$$

Clear, $Q(x) > 0$ for $x \in (0, 1/20]$. While $P(x) < 0$ for $x \in (0, 1/20]$ due to $P'(x) > 0$ and so

$$P(x) \leq P\left(\frac{1}{20}\right) = -\frac{2874\,530\,403\,954\,909\,124\,821}{1024\,000\,000\,000} < 0,$$

which leads to $F''_{a'_0}(x) < 0$ for $x \in (0, 1/20]$.

This completes the proof. \square

Theorem 3. *Let the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ be defined on $(0, \infty)$ in which $\mathcal{L}(x, a)$ be defined by 1.8. Then F_a is convex on $(0, \infty)$ if and only if $a \in (1/15, a''_0]$, where $a''_0 \approx 0.4321803644583305$ is the unique root of the equation*

$$F''_a(0) = \psi''(1) - \mathcal{L}_{xx}(0, a) = 0$$

on $(1/15, \infty)$, here $\mathcal{L}_{xx}(x, a)$ is defined by (2.4).

Proof. Necessity. The necessity is deduced from

$$\begin{aligned} F''_a(0) &= \psi''(1) - \mathcal{L}_{xx}(0, a) \\ &= \psi''(1) - \frac{a - \frac{1}{6}}{(45a^2 + 1)\left(a + \frac{1}{3}\right)^2} + \frac{2025}{2} \frac{a^3(15a + 2)}{(15a - 1)^2(45a^2 + 1)} \\ &= \psi''(1) + \frac{3}{2} \frac{2025a^4 + 1620a^3 + 360a^2 - 36a + 1}{(3a + 1)^2(15a - 1)^2} \geq 0. \end{aligned}$$

Since $a \mapsto \mathcal{L}_{xx}(x, a)$ is increasing on $(1/15, \infty)$, so $a \mapsto F''_a(0)$ is decreasing on the same interval. Note that the facts

$$F''_{1/3}(0) = \frac{171}{64} - 2\zeta(3) \approx 0.26776 > 0 \text{ and } F''_{1/2}(0) = \frac{19\,299}{8450} - 2\zeta(3) \approx -0.12021 < 0,$$

we see that the equation $F''_a(0) = 0$ has a unique solution $a''_0 \in (1/3, 1/2)$ such that $F''_a(0) > 0$ for $a \in (1/15, a''_0)$ and $F''_a(0) < 0$ for $a \in (a''_0, \infty)$. Therefore, the solution of the inequality $F''_a(0) \geq 0$ is $a \in (1/15, a''_0]$. Numerical calculation gives $a''_0 \approx 0.4321803644583305$.

Sufficiency. Now we prove the condition $a \in (1/15, a''_0]$ is sufficient for $F''_a(x) > 0$ to hold for $x \in (0, \infty)$. Due to the increasing property of $\mathcal{L}_{xx}(x, a)$ with respect to a shown by Lemma 1, we only need to prove $F''_{a''_0}(x) > 0$. We distinguish to cases:

Case 1: $x \in (3/50, \infty)$. Using (2.7) and Remark 1 together with $a_0'' < 9/20$, we have

$$\begin{aligned} F_a''(x+1) - F_a''(x) &= \psi''(x+2) - \psi''(x+1) - \mathcal{L}_{xx}(x+1, a) + \mathcal{L}_{xx}(x, a) \\ &= \frac{2}{(x+1)^3} - (\mathcal{L}_{xx}(x+1, 9/20) - \mathcal{L}_{xx}(x, 9/20)) := -2 \frac{r(x)}{s(x)}, \end{aligned}$$

where

$$\begin{aligned} r(x) &= 125\,413\,273\,555\,200x^{10} + 1254\,132\,735\,552\,000x^9 + 5518\,250\,043\,762\,960x^8 \\ &\quad + 14\,046\,814\,696\,855\,680x^7 + 22\,840\,386\,490\,946\,664x^6 + 24\,664\,633\,018\,794\,864x^5 \\ &\quad + 17\,718\,225\,566\,437\,953x^4 + 8120\,232\,997\,769\,412x^3 + 2081\,281\,129\,927\,908x^2 \\ &\quad + 179\,154\,971\,702\,976x - 19\,953\,618\,766\,474, \end{aligned}$$

$$\begin{aligned} s(x) &= (60x + 60x^2 + 47)^2 (81x + 81x^2 + 23)^2 \\ &\quad \times (180x + 60x^2 + 167)^2 (243x + 81x^2 + 185)^2 (x+1)^3. \end{aligned}$$

Since $r'(x) > 0$, we get

$$r(x) > r(3/50) = \frac{1114\,560\,148\,894\,087\,067\,992\,508}{3814\,697\,265\,625} > 0$$

for $x \in (3/50, \infty)$, while $s(x)$ is obviously positive on the same interval. It follows that $F_a''(x+1) - F_a''(x) < 0$ for $x \in (1/10, \infty)$, and therefore, $F_a''(x) > \lim_{x \rightarrow \infty} F_a''(x) = 0$ for $x \in (1/10, \infty)$.

Case 2: $x \in (0, 3/50]$. If we show that $F_{a_0}'''(x) > 0$ for $x \in (0, 3/50]$, then we get $F_{a_0}''(x) \geq F_{a_0}''(0) = 0$, which proves the desired result. Now replacing x by $x+3/2$ in (3.11) and using (2.7), we have

$$\begin{aligned} \psi'''(x+1) &> \frac{6}{(x+1)^4} + \frac{160}{3}(x+3/2) \\ &\quad \times \frac{6585600(x+3/2)^8 + 15052800(x+3/2)^6 + 11696160(x+3/2)^4 + 1820960(x+3/2)^2 - 701703}{(560(x+3/2)^4 + 520(x+3/2)^2 + 27)^3}. \end{aligned}$$

In view of $a \mapsto \mathcal{L}_{xxx}(x, a)$ is decreasing by Lemma 1 and $a_0'' \approx 0.432180 > 21/50$, we get

$$\mathcal{L}_{xxx}(x, a_0'') < \mathcal{L}_{xxx}\left(x, \frac{21}{50}\right) = \frac{3969}{4469} \frac{(2x+1)(x^2+x+\frac{10}{63})}{(x^2+x+\frac{53}{189})^3} + \frac{500}{4469} \frac{(2x+1)(x^2+x-\frac{63}{50})}{(x^2+x+\frac{113}{150})^3}.$$

Then we have

$$\begin{aligned} F_{a_0}'''(x) &= \psi'''(x+1) - \mathcal{L}_{xx}(x, a_0'') \\ &> \frac{6}{(x+1)^4} + \frac{160}{3}(x+3/2) \times \\ &\quad \frac{6585600(x+3/2)^8 + 15052800(x+3/2)^6 + 11696160(x+3/2)^4 + 1820960(x+3/2)^2 - 701703}{(560(x+3/2)^4 + 520(x+3/2)^2 + 27)^3} \\ &\quad - \frac{3969}{4469} \frac{(2x+1)(x^2+x+\frac{10}{63})}{(x^2+x+\frac{53}{189})^3} - \frac{500}{4469} \frac{(2x+1)(x^2+x-\frac{63}{50})}{(x^2+x+\frac{113}{150})^3}. \end{aligned}$$

Factoring and arranging lead to

$$F_{a_0}'''(x) > -\frac{1}{3} \frac{R(x)}{S(x)},$$

where

$$\begin{aligned}
R(x) = & 1439\,970\,288\,529\,500\,000x^{19} + 33\,839\,301\,780\,443\,250\,000x^{18} \\
& + 377\,685\,219\,317\,959\,507\,500x^{17} + 2619\,038\,198\,507\,995\,293\,750x^{16} \\
& + 12\,578\,516\,662\,166\,748\,200\,250x^{15} + 44\,394\,499\,254\,715\,419\,844\,125x^{14} \\
& + 119\,436\,801\,689\,614\,664\,479\,875x^{13} + 250\,817\,342\,412\,016\,626\,059\,625x^{12} \\
& + 417\,457\,335\,039\,758\,233\,395\,000x^{11} + 555\,642\,395\,442\,917\,892\,895\,800x^{10} \\
& + 593\,602\,907\,219\,352\,981\,396\,390x^9 + 508\,233\,654\,389\,427\,279\,197\,745x^8 \\
& + 346\,198\,219\,129\,731\,218\,829\,124x^7 + 184\,849\,155\,080\,550\,188\,733\,310x^6 \\
& + 75\,353\,569\,007\,634\,565\,613\,769x^5 + 22\,380\,430\,314\,381\,942\,509\,812x^4 \\
& + 4414\,609\,088\,286\,249\,144\,994x^3 + 450\,421\,073\,304\,504\,390\,873x^2 \\
& - 4721\,565\,008\,851\,422\,102x - 4420\,688\,040\,144\,642\,816,
\end{aligned}$$

$$\begin{aligned}
S(x) = & (x+1)^4 (150x^2 + 150x + 113)^3 (189x^2 + 189x + 53)^3 \\
& \times (35x^4 + 210x^3 + 505x^2 + 570x + 252)^3.
\end{aligned}$$

A simple computation gives $R''(x) > 0$ and

$$\begin{aligned}
R(0) &= -4420\,688\,040\,144\,642\,816 < 0, \\
R\left(\frac{3}{50}\right) &= -\frac{337\,711\,343\,455\,989\,855\,048\,292\,675\,691\,209\,992\,531\,618\,111}{190\,734\,863\,281\,250\,000\,000\,000\,000} < 0,
\end{aligned}$$

which by property of convex functions yield that for $x \in (0, 3/50]$,

$$R(x) \leq \frac{3/50 - x}{3/50} R(0) + \frac{x}{3/50} R\left(\frac{3}{50}\right) < 0.$$

This in combination with $S(x) > 0$ gives $F_{a_0}'''(x) > 0$ for $x \in (0, 3/50]$.

Thus we complete the proof. \square

As a direct consequence of Theorems 2 and 3, we have

Corollary 2. *Let the function $x \rightarrow F_a(x) = \psi(x+1) - \mathcal{L}(x, a)$ be defined on $(0, \infty)$ where $\mathcal{L}(x, a)$ be defined by 1.8. Then for $x > 0$, F_a is decreasing and convex if and only if $a \in (1/15, a_0'')$, where $a_0'' \approx 0.4321803644583305$ is defined in 3.*

A easy computation gives

$$\begin{aligned}
\mathcal{L}_x(x, a_1) &= \left(x + \frac{1}{2}\right) \frac{x+x^2+23/21}{x^4+2x^3+17x^2/7+10x/7+12/35}, \\
\mathcal{L}_x(x, a_0') &= \left(x + \frac{1}{2}\right) \frac{x^2+x+\frac{\pi^2}{15(\pi^2-9)}}{x^4+2x^3+\frac{7\pi^2-60}{5(\pi^2-9)}x^2+\frac{2\pi^2-15}{5(\pi^2-9)}x+\frac{1}{5(\pi^2-9)}},
\end{aligned}$$

and by Corollary 1 and Theorem 2 we obtain the following optimal inequalities.

Corollary 3. *For $x > 0$, the double inequality*

$$\mathcal{L}_x(x, a_1) < \psi'(x+1) < \mathcal{L}_x(x, a_0')$$

or equivalently,

$$\left(x + \frac{1}{2}\right) \frac{x+x^2+\frac{23}{21}}{x^4+2x^3+\frac{17}{7}x^2+\frac{10}{7}x+\frac{12}{35}} < \psi'(x+1) < \left(x + \frac{1}{2}\right) \frac{x^2+x+\frac{\pi^2}{15(\pi^2-9)}}{x^4+2x^3+\frac{7\pi^2-60}{5(\pi^2-9)}x^2+\frac{2\pi^2-15}{5(\pi^2-9)}x+\frac{1}{5(\pi^2-9)}}$$

holds with the best constants

$$a_1 = \frac{40+3\sqrt{205}}{105} \approx 0.79003 \text{ and } a_0' = \frac{45-4\pi^2+3\sqrt{4\pi^4-80\pi^2+405}}{30(\pi^2-9)} \approx 0.47053.$$

Similarly, from Corollary 1 and Theorem 3 we obtain

Corollary 4. *For $x > 0$, the double inequality*

$$\mathcal{L}_{xx}(x, a_0'') < \psi''(x+1) < \mathcal{L}_{xx}(x, a_1)$$

holds with the best constants $a_0'' \approx 0.4321803644583305$ and $a_1 = (40 + 3\sqrt{205})/105 \approx 0.79003$.

Particularly, taking $a = 1/3 < a_0''$, we have

$$\begin{aligned} & -\frac{9}{2} \frac{450x^6 + 1350x^5 + 1965x^4 + 1680x^3 + 897x^2 + 282x + 38}{(3x^2 + 3x + 2)^2(15x^2 + 15x + 4)^2} \\ & < \psi''(x+1) < -\frac{5}{6} \frac{(1470x^6 + 4410x^5 + 7875x^4 + 8400x^3 + 5863x^2 + 2398x + 346)}{(35x^4 + 70x^3 + 85x^2 + 50x + 12)^2} \end{aligned}$$

4. SHARP BOUNDS FOR PSI FUNCTION AND THE HARMONIC NUMBER

Theorem 4. *Let the function $x \rightarrow \mathcal{L}(x, a)$ be defined on $(-1, \infty)$ by (1.8) and $a \in (4/15, \infty)$. Then inequality*

$$(4.1) \quad \psi(x+1) < \mathcal{L}(x, a)$$

holds for all $x \in (-1, \infty)$ if and only if $a \geq a_1 = (40 + 3\sqrt{205})/105 \approx 0.79003$.

Moreover, for $x > 0$ we have

$$(4.2) \quad \mathcal{L}(x, a) - c_0(a) < \psi(x+1) < \mathcal{L}(x, a),$$

where

$$(4.3) \quad c_0(a) = \mathcal{L}(0, a) + \gamma = \frac{1}{90a^2+2} \ln \frac{3a+1}{3} + \frac{45a^2}{90a^2+2} \ln \frac{15a-1}{45a} + \gamma$$

is the best constant, and the lower bound $\mathcal{L}(x, a) - c_0(a)$ and upper bound $\mathcal{L}(x, a)$ are decreasing and increasing on (a_1, ∞) , respectively.

Proof. Necessity. If (4.1) holds, that is, $F_a(x) = \psi(x+1) - \mathcal{L}(x, a) < 0$, then by (1.11) we have

$$\lim_{x \rightarrow \infty} \frac{F_a(x)}{x^{-6}} = -\frac{1}{85050a} \left(a - \frac{40+3\sqrt{205}}{105} \right) \left(a - \frac{40-3\sqrt{205}}{105} \right) \leq 0.$$

Solving the inequality for a and noting that $a \in (4/15, \infty)$ yield

$$a \geq \frac{40+3\sqrt{205}}{105} = a_1,$$

which shows that the condition $a \geq a_1$ is necessary.

Sufficiency. Suppose that $a \geq a_1$. By Theorem 1, it is deduced that

$$F_{a_1}(x) = \psi(x+1) - \mathcal{L}(x, a_1) < 0,$$

that is, $\psi(x+1) < \mathcal{L}(x, a_1)$ holds for all $x \in (-1, \infty)$. Since the function $a \rightarrow \mathcal{L}(x, a)$ is increasing on $(4/15, \infty)$ by Lemma 1, it is easy to conclude that for $a \geq a_1$,

$$\psi(x+1) < \mathcal{L}(x, a_1) \leq \mathcal{L}(x, a)$$

holds for all $x \in (-1, \infty)$, which means that the condition $a \geq a_0$ is sufficient.

Using the monotonicity of $F_a(x)$ and the facts $F_a(0) = -\gamma - \mathcal{L}(0, a)$ and $F_a(\infty) = 0$ gives 4.2, and the monotonicity of the lower and upper bounds in a follows from Lemma 1 and Remark 1.

This completes the proof. \square

Letting $a = 4/5, 1, \infty$ in Theorem 4 we have

Corollary 5. *The following double inequalities*

$$\begin{aligned}
& \frac{5}{298} \ln \left(x^2 + x + \frac{17}{15} \right) + \frac{72}{149} \ln \left(x^2 + x + \frac{11}{36} \right) - c_0(4/5) \\
& < \psi(x+1) < \frac{5}{298} \ln \left(x^2 + x + \frac{17}{15} \right) + \frac{72}{149} \ln \left(x^2 + x + \frac{11}{36} \right), \\
& \frac{1}{92} \ln \left(x^2 + x + \frac{4}{3} \right) + \frac{45}{92} \ln \left(x^2 + x + \frac{14}{45} \right) - c_0(1) \\
& < \psi(x+1) < \frac{1}{92} \ln \left(x^2 + x + \frac{4}{3} \right) + \frac{45}{92} \ln \left(x^2 + x + \frac{14}{45} \right), \\
(4.4) \quad & \frac{1}{2} \ln \left(x^2 + x + \frac{1}{3} \right) - c_0(\infty) < \psi(x+1) < \frac{1}{2} \ln \left(x^2 + x + \frac{1}{3} \right),
\end{aligned}$$

hold true for $x > 0$, where

$$\begin{aligned}
c_0(4/5) &= \gamma + \frac{5}{298} \ln \frac{17}{15} + \frac{72}{149} \ln \frac{11}{36} \approx 0.0063957, \\
c_0(1) &= \gamma + \frac{1}{92} \ln \frac{4}{3} + \frac{45}{92} \ln \frac{14}{45} \approx 0.0092314, \\
c_0(\infty) &= \gamma + \lim_{a \rightarrow \infty} \left(\frac{1}{90a^2+2} \ln \frac{3a+1}{3} + \frac{45a^2}{90a^2+2} \ln \frac{15a-1}{45a} \right) = \gamma - \frac{1}{2} \ln 3 \approx 0.027910
\end{aligned}$$

are the best constants.

Remark 4. We easily check that the lower bound in (4.4) is weaker than one in (1.5).

By the relation $\psi(n+1) = H_n - \gamma$ and the fact $F_a(1) = 1 - \gamma - \mathcal{L}(1, a)$, the inequalities 4.2 can be changed into

Corollary 6. Let $\mathcal{L}(x, a)$ be defined by (1.8) and $a \geq a_1 = (40 + 3\sqrt{205})/105$. Then for all $n \in \mathbb{N}$ we have

$$(4.5) \quad \mathcal{L}(n, a) + c_1(a) < H_n < \mathcal{L}(n, a) + \gamma,$$

where $c_1(a) = 1 - \mathcal{L}(1, a)$ and γ are the best possible. And, the lower bound $\mathcal{L}(n, a) + c_1(a)$ and upper bound $\mathcal{L}(n, a) + \gamma$ are decreasing and increasing on (a_1, ∞) , respectively.

Theorem 5. Let the function $x \rightarrow \mathcal{L}(x, a)$ be defined on $(0, \infty)$ by (1.8). Then inequality

$$(4.6) \quad \psi(x+1) > \mathcal{L}(x, a)$$

holds for all $x > 0$ if and only if $a \in (1/15, a_0]$, where $a_0 \approx 0.512967071402$ is the unique root of the equation $F_a(0) = \psi(1) - \mathcal{L}(0, a) = 0$ on $(1/15, \infty)$.

Proof. Necessity. The necessity can be derived from $F_a(0) = \psi(1) - \mathcal{L}(0, a) \geq 0$. Lemma 1 shows that the function \mathcal{L} is increasing with a on $(1/15, \infty)$, which implies that the function $a \rightarrow F_a(0) = \psi(1) - \mathcal{L}(0, a)$ is decreasing on $(1/15, \infty)$. Straightforward computations yield

$$F_{1/2}(0) = 4.0043 \times 10^{-4} > 0 \quad \text{and} \quad F_{3/5}(0) = -2.3727 \times 10^{-3} < 0,$$

which reveals that there is a unique point $a_0 \in (1/15, 3/5)$ satisfying $F_{a_0}(0) = 0$ such that $F_a(0) > 0$ for $a \in (1/15, a_0)$ and $F_a(0) < 0$ for $a \in (a_0, \infty)$.

Numerical calculation gives $a_0 \approx 0.512967071402$, which shows the necessity.

Sufficiency. By part two of Lemma 1, to prove sufficiency, it is enough to show that

$$F_{a_0}(x) = \psi(x+1) - \mathcal{L}(x, a_0) \geq 0$$

holds for all $x > 0$. Now we prove it stepwise.

(i) First of all, we prove that $F'_{a_0}(x) < 0$ for $x \geq 1/5$. Since $\lim_{x \rightarrow \infty} F'_{a_0}(x) = 0$, from Lemma 3, it suffices to show that

$$F'_{a_0}(x+1) - F'_{a_0}(x) = \frac{q(x, a_0)}{p(x, a_0)} > 0,$$

where $q(x, a)$, $p(x, a)$ are defined by (3.6) and (3.7), respectively.

Since function $a \rightarrow q(x, a)$ is decreasing on $(1/15, \infty)$ by (3.8) and $a_0 < 11/21$, for $x \geq 1/5$, we get

$$\begin{aligned} q(x, a_0) &> q(x, \frac{11}{21}) = \left[\frac{-315a^2+240a+7}{2025a} (x+1)^2 - \frac{(a+1/3)^2(a-1/15)^2}{9a^2} \right]_{a=11/21} \\ &= \frac{12}{275} (x+1)^2 - \frac{9216}{148225} \geq \frac{12}{275} (1/5+1)^2 - \frac{9216}{148225} = \frac{2448}{3705625} > 0, \end{aligned}$$

which together with $p(x, a_0) > 0$ yields $F'_{a_0}(x+1) - F'_{a_0}(x) > 0$. By Lemma 3, we have $F'_{a_0}(x) < 0$ for $x \geq 1/5$.

(ii) Secondly, we show that there is a point $x_0 \in (0, 1/5)$ such that $F'_{a_0}(x_0) = 0$, and $F'_{a_0}(x) > 0$ if $x \in (0, x_0)$ and $F'_{a_0}(x) < 0$ if $x \in (x_0, 1/5)$. For this purpose, it suffice to prove that F'_{a_0} is decreasing on $(0, 1/5)$ and $F'_{a_0}(0) > 0$.

Replacing x by $x+3/2$ in (3.10) and using (2.7), we have

$$(4.7) \quad \psi''(x+1) < -\frac{20}{3} \frac{47040(x+3/2)^6 + 75600(x+3/2)^4 + 30116(x+3/2)^2 - 1917}{(560(x+3/2)^4 + 520(x+3/2)^2 + 27)^2} - \frac{2}{(x+1)^3}.$$

Since $a \mapsto \mathcal{L}_{xx}(x, a_0)$ is increasing by Lemma 1 and $a_0 > 1/2$, we get

$$(4.8) \quad \mathcal{L}_{xx}(x, a_0) > \mathcal{L}_{xx}(x, \frac{1}{2}) = -\frac{4}{49} \frac{x^2+x-1/3}{(x^2+x+5/6)^2} - \frac{45}{49} \frac{x^2+x+19/90}{(x^2+x+13/45)^2},$$

where $\mathcal{L}_{xx}(x, a)$ is given by (2.4). Utilizations of (4.7) and (4.8) yield that

$$\begin{aligned} F''_{a_0}(x) &= \psi''(x+1) - \mathcal{L}_{xx}(x, a_0) \\ &< -\frac{20}{3} \frac{47040(x+1+1/2)^6 + 75600(x+1+1/2)^4 + 30116(x+1+1/2)^2 - 1917}{(560(x+1+1/2)^4 + 520(x+1+1/2)^2 + 27)^2} \\ &\quad - \frac{2}{(x+1)^3} - \mathcal{L}_{xx}(x, \frac{1}{2}) \\ &= -\frac{20}{3} \frac{47040(x+1+1/2)^6 + 75600(x+1+1/2)^4 + 30116(x+1+1/2)^2 - 1917}{(560(x+1+1/2)^4 + 520(x+1+1/2)^2 + 27)^2} - \frac{2}{(x+1)^3} \\ &\quad + \frac{4}{49} \frac{x^2+x-1/3}{(x^2+x+5/6)^2} + \frac{45}{49} \frac{x^2+x+19/90}{(x^2+x+13/45)^2}. \end{aligned}$$

Factoring and arranging lead to

$$F''_{a_0}(x) < \frac{32}{54675} \frac{v(x)}{u(x)},$$

where

$$\begin{aligned} u(x) &= \left(560 \left(x + \frac{3}{2} \right)^4 + 520 \left(x + \frac{3}{2} \right)^2 + 27 \right)^2 (x+1)^3 \\ &\quad \times \left(x^2 + x + \frac{13}{45} \right)^2 \left(x^2 + x + \frac{5}{6} \right)^2 > 0, \end{aligned}$$

$$\begin{aligned} v(x) &= 25533900x^{11} + 383008500x^{10} + 2632741950x^9 + 10267850400x^8 \\ &\quad + 24970713315x^7 + 39608501505x^6 + 41428932346x^5 + 27795216042x^4 \\ &\quad + 10641326265x^3 + 1197017371x^2 - 633162120x - 192808962. \end{aligned}$$

A simple computation yields $v''(x) > 0$ for $x > 0$ and

$$v(0) = -192808962 < 0 \quad \text{and} \quad v(1/5) = -\frac{245738739045744}{1953125} < 0,$$

which by properties of convex functions reveals that for $x \in (0, 1/5)$

$$v(x) < (1 - 5x) \times v(0) + 5x \times v\left(\frac{1}{5}\right) < 0.$$

Thus, $F''_{a_0}(x) < 0$ for $x \in (0, 1/5)$, which means that $F'_{a_0}(x)$ is decreasing on $(0, 1/5)$.

On the other hand, note that $\mathcal{L}_x(x, a)$ decreases with a on $(1/15, \infty)$ by Lemma 1 and $a_0 > 1/2$, it is derived that

$$\begin{aligned} F'_{a_0}(0) &= \psi'(1) - \mathcal{L}_x(0, a_0) > \psi'(1) - \mathcal{L}_x(0, \tfrac{1}{2}) \\ &= \frac{\pi^2}{6} - \frac{213}{130} \approx 6.4725 \times 10^{-3} > 0. \end{aligned}$$

This together with the assertion that $F'_{a_0}(x) < 0$ for $x \geq 1/5$ proved previously implies that there is a unique point $x_0 \in (0, 1/5)$ such that $F'_{a_0}(x_0) = 0$, and $F'_{a_0}(x) > 0$ for $x \in (0, x_0)$ and $F'_{a_0}(x) < 0$ for $x \in (x_0, 1/5)$.

(iii) Finally, from (i) and (ii) we conclude that $F'_{a_0}(x) > 0$ for $x \in (0, x_0)$ and $F'_{a_0}(x) < 0$ for $x \in (x_0, \infty)$. It follows that

$$\begin{aligned} F_{a_0}(x_0) &> F_{a_0}(x) > F_{a_0}(0) = 0 \text{ for } x \in (0, x_0), \\ F_{a_0}(x_0) &\geq F_{a_0}(x) > \lim_{x \rightarrow \infty} F_{a_0}(x) = 0 \text{ for } x \in [x_0, \infty), \end{aligned}$$

where $F_{a_0}(0) = 0$ is due to a_0 is the unique root of the equation $F_{a_0}(0) = 0$.

This completes the proof. \square

Remark 5. From the proof previously we see that $F_{a_0}(x)$ has an upper bound $F_{a_0}(x_0)$ in which x_0 is a unique zero point of $F'_{a_0}(x)$. Numerical computation yields

$$x_0 \approx 0.147311876217, \quad F_{a_0}(x_0) \approx 0.0004651.$$

It follows that

$$(4.9) \quad \mathcal{L}(x, a_0) \leq \psi(x+1) \leq \mathcal{L}(x, a_0) + 0.0004651$$

holds for all $x > 0$.

By the proof previously and Lemma 1, we easily obtain that F_a is decreasing on $[1/5, \infty)$ for $a \in (1/15, a_0)$. It follows from the relation $\psi(n+1) = H_n - \gamma$ with the fact $F_a(1) = 1 - \gamma - \mathcal{L}(1, a)$ that

Corollary 7. Let $\mathcal{L}(x, a)$ be defined by (1.8) and $a_0 = 0.512967071402\dots$. Then

$$(4.10) \quad \mathcal{L}(n, a) + \gamma < H_n < \mathcal{L}(n, a) + c_1(a),$$

holds for all $n \in \mathbb{N}$ and $a \in (1/15, a_0)$, where $c_1(a) = 1 - \mathcal{L}(1, a)$ and γ are the best.

In particular, taking $a = 1/2$, we have

$$\begin{aligned} &\frac{2}{49} \ln \left(n^2 + n + \frac{5}{6} \right) + \frac{45}{98} \ln \left(n^2 + n + \frac{13}{45} \right) + \gamma \\ &< H_n < \frac{2}{49} \ln \left(n^2 + n + \frac{5}{6} \right) + \frac{45}{98} \ln \left(n^2 + n + \frac{13}{45} \right) + c_1(1/2), \end{aligned}$$

where

$$c_1(1/2) = 1 - \frac{2}{49} \ln \frac{17}{6} - \ln \frac{103}{45} \approx 0.57726.$$

For $a \in (1/15, a'_0]$, utilizing the decreasing property of F_a on $(0, \infty)$ together with the facts $F_a(0) = -\gamma - \mathcal{L}(0, a)$ and $F_a(\infty) = 0$, we have

Corollary 8. *For $x > 0$, the double inequality*

$$(4.11) \quad \mathcal{L}(x, a) < \psi(x+1) < \mathcal{L}(x, a) - c_0(a),$$

where $c_0(a)$ defined by (4.3) is the best constant. And, the lower bound $\mathcal{L}(x, a)$ and upper bound $\mathcal{L}(x, a) - c_0(a)$ are respectively increasing and decreasing on $(1/15, a'_0]$, where $a'_0 \approx 0.47053$ is defined by (3.12). Particularly, taking $a = 1/3, 4/15, \sqrt{5}/15, 1/15^+$, we have

$$(4.12) \quad \begin{aligned} & \frac{1}{12} \ln \left(x^2 + x + \frac{2}{3} \right) + \frac{5}{12} \ln \left(x^2 + x + \frac{4}{15} \right) \\ < \psi(x+1) < \frac{1}{12} \ln \left(x^2 + x + \frac{2}{3} \right) + \frac{5}{12} \ln \left(x^2 + x + \frac{4}{15} \right) - c_0(1/3), \end{aligned}$$

$$(4.12) \quad \begin{aligned} & \frac{16}{21} \ln \left(x + \frac{1}{2} \right) + \frac{5}{42} \ln \left(x^2 + x + \frac{3}{5} \right) \\ < \psi(x+1) < \frac{16}{21} \ln \left(x + \frac{1}{2} \right) + \frac{5}{42} \ln \left(x^2 + x + \frac{3}{5} \right) - c_0(4/15), \end{aligned}$$

$$(4.13) \quad \begin{aligned} & \frac{1}{4} \ln \left(\left(x^2 + x + \frac{1}{3} \right)^2 - \frac{1}{45} \right) \\ < \psi(x+1) < \frac{1}{4} \ln \left(\left(x^2 + x + \frac{1}{3} \right)^2 - \frac{1}{45} \right) - c_0(\sqrt{5}/15), \end{aligned}$$

$$(4.14) \quad \psi(x+1) > \frac{1}{12} \ln(x^2 + x) + \frac{5}{12} \ln \left(x^2 + x + \frac{2}{5} \right),$$

where

$$\begin{aligned} c_0(1/3) &= \frac{1}{12} \ln \frac{2}{3} + \frac{5}{12} \ln \frac{4}{15} + \gamma \approx -0.0073047, \\ c_0(4/15) &= -\frac{8}{21} \ln 4 + \frac{5}{42} \ln \frac{3}{5} + \gamma \approx -0.011709, \\ c_0(\sqrt{5}/15) &= \frac{1}{4} \ln \frac{4}{45} + \gamma \approx -0.027876. \end{aligned}$$

Remark 6. *The lower bound for $\psi(x+1)$ in (4.12) is clearly stronger than one in (1.2).*

5. APPROXIMATIONS OF EULER'S CONSTANT

The Euler's constant γ defined by the limit relation

$$\gamma = \lim_{n \rightarrow \infty} (H_n - \ln n) = 0.577215664...,$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the n 'th harmonic number, is one of the most important constants in mathematics, maybe the third next to π and e . It is known that the classical sequence $\gamma_n = H_n - \log n$ converges to γ very slowly. In fact, Young [40]

proved that the sequence (γ_n) converges to γ as n^{-1} . As a consequence, many mathematicians tried to define new sequences convergent to this constant with increasingly higher speed. For example, DeTemple [22], [21] introduced a faster sequence $T_n = H_n - \log(n + 1/2)$ and showed that the sequence (T_n) convergent to γ like n^{-2} . In [38], Toth defined the sequence

$$T_n = H_n - \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right),$$

which converges to γ as n^{-3} proved by Negoï in [35]. Batir [11] gave a sequence (μ_n) defined by

$$\mu_n = H_n + \frac{1}{2} \ln \frac{e^{1/(n+1)} - 1}{n + 1/2},$$

which converges to γ as n^{-3} proved by Motici in [31]. In a very recent papers [26], [27], [31], [32], [34], [33], [28], [30], C. Mortici established several new sequences converge to γ at faster rate, for instance, he [30] proved that the sequences (u_n) and (v_n) defined by, respectively,

$$\begin{aligned} u_n &= H_{n-1} + \frac{1}{(6 - 2\sqrt{6})n} - \ln \left(n + 1/\sqrt{6} \right), \\ v_n &= H_{n-1} + \frac{1}{(6 + 2\sqrt{6})n} - \ln \left(n - 1/\sqrt{6} \right) \end{aligned}$$

converge to γ as n^{-3} , and (δ_n) converge to γ as n^{-4} , where δ_n is the arithmetic mean between u_n and v_n . Another sequence converges to γ as n^{-4} is (α_n) defined by

$$\alpha_n = H_{n-2} + \frac{23}{24(n-1)} + \frac{1}{24n} - \ln(n - 1/2)$$

given in [30]. He also gave a sequence that speed of convergence is n^{-5} in [32], however, it is complicated.

In [15] Batir proposed two better estimations for γ : (σ_n) defined by (1.7) and (θ_n) defined by

$$\theta_n = H_n + \frac{1}{2} \ln \frac{e^{2/(n+1)} - 1}{2n + 2},$$

which also converge to γ as n^{-4} . Furthermore, he defined (τ_n) as

$$\tau_n = \frac{\theta_n + \sigma_n}{2} = H_n + \frac{1}{4} \ln \frac{e^{2/(n+1)} - 1}{2n^3 + 4n^2 + 8n/3 + 2/3}$$

and pointed out that the sequence (τ_n) converges to γ at rate faster than n^{-4} without proving. Indeed, it is not difficult to show that

$$\lim_{n \rightarrow \infty} n^5 (\tau_n - \gamma) = -\frac{1}{180},$$

which implies that (τ_n) converges to γ like n^{-5} . So far, this is one of the best results.

Now we consider our results in this paper. For $a \in (1/105, \infty)$, we define a sequence with a parameter $(l_n(a))$ as

$$(5.1) \quad l_n(a) = H_n - \frac{1}{90a^2+2} \ln \left(n^2 + n + \frac{3a+1}{3} \right) - \frac{45a^2}{90a^2+2} \ln \left(n^2 + n + \frac{15a-1}{45a} \right)$$

From (2.1) it is easy to get

$$l_n(\infty) = H_n - \frac{1}{2} \ln(n^2 + n + \frac{1}{3}) = \sigma_n.$$

By the relation $\psi(n+1) = H_n - \gamma$ we have

$$(5.2) \quad l_n(a) - \gamma = \psi(n+1) - \mathcal{L}(n, a).$$

Thus the limit relations (1.11) and (1.12) can be written as

$$\begin{aligned} \lim_{n \rightarrow \infty} n^6 (l_n(a) - \gamma) &= -\frac{1}{85050a} \left(a - \frac{40+3\sqrt{205}}{105} \right) \left(a - \frac{40-3\sqrt{205}}{105} \right), \\ \lim_{n \rightarrow \infty} n^8 (l_n(a_1) - \gamma) &= -\frac{2}{1225}. \end{aligned}$$

These show that for every $a \in (1/105, \infty)$ the sequence $(l_n(a))$ converges to γ as n^{-6} if $a \neq a_1$ and as n^{-8} if $a = a_1$.

Not only that, from Corollaries 7 and 6, we have

Theorem 6. *Let the sequence $(l_n(a))$ be defined by (5.1) where $a \in (1/15, \infty)$. Then for all $n \in \mathbb{N}$,*

$$l_n(a_1) < \gamma \leq l_n(a_0)$$

hold, where $a_1 = (40 + 3\sqrt{205})/105$ is the best constant, $a_0 \approx 0.512967071402$ is defined in Theorem 5. Also, for every $n \in \mathbb{N}$, $l_n(a)$ is strictly decreasing with a on $(1/15, \infty)$.

It is clear that our sequence $(l_n(a))$ defined by (5.1) gives very accurate values for γ than the approximations mentioned above, which also can be seen in the following table (for convenience, we take $a = 1/2 < a_0$).

n	$ \delta_n - \gamma $	$ \tau_n - \gamma $	$ l_n(a_1) - \gamma $	$ l_n(1/2) - \gamma $
1	1.3945×10^{-2}	2.8251×10^{-4}	2.1784×10^{-5}	4.1397×10^{-5}
2	9.1696×10^{-4}	3.2546×10^{-5}	6.6758×10^{-7}	3.2255×10^{-6}
5	2.425×10^{-5}	8.6636×10^{-7}	1.7431×10^{-9}	3.7717×10^{-8}
10	1.5246×10^{-6}	3.8479×10^{-8}	1.0704×10^{-11}	8.2711×10^{-10}
50	2.4442×10^{-9}	1.6499×10^{-11}	3.8544×10^{-17}	6.8338×10^{-14}
100	1.5277×10^{-10}	5.3517×10^{-13}	1.5682×10^{-19}	1.1009×10^{-15}
200	9.5486×10^{-12}	1.7039×10^{-14}	6.2509×10^{-22}	1.7464×10^{-17}
500	2.4444×10^{-13}	1.7645×10^{-16}	4.1430×10^{-25}	7.2181×10^{-20}

Furthermore, from Corollaries 1 and 2, we see that our sequence $(l_n(a))$ has well properties, such as monotonicity and concavity, which are stated as follows.

Theorem 7. *Let $n \in \mathbb{N}$. Then the sequence $(l_n(a))$ is strictly increasing and concave if $a \geq a_1 = (40 + 3\sqrt{205})/105$, and decreasing and convex if $a \in (1/15, a_0'')$, where $a_0'' \approx 0.4321803644583305$ is defined in 3..*

6. OPEN PROBLEMS

Motivated by Theorem 1, we post the following open problems.

Problem 1. *Let $\mathcal{L}(x, a)$ be defined by 1.8. Prove that $-F_{a_1}(x) = \mathcal{L}(x, a_1) - \psi(x+1)$ is a completely monotonic function on $(-1, \infty)$.*

REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1972.
- [2] H. Alzer, Some gamma function inequalities, *Math. Comp.*, **60** (201) (1993) 337–346.
- [3] H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comp.*, **66** (217) (1997) 373–389.
- [4] H. Alzer AND J. Wells, Inequalities for the polygamma functions, *SIAM J. Math. Anal.*, **29** (6) (1998) 1459–1466.
- [5] H. Alzer, Inequalities for the Gamma function, *Proc. Amer. Math. Soc.*, **128** (1) (1999) 141–147.
- [6] H. Alzer AND S. Ruscheweyh, A subadditive property of the gamma function, *J. Math. Anal. Appl.*, **285** (2003) 564–577.
- [7] H. Alzer, Sharp inequalities for digamma and polygamma functions, *Forum Math.*, **16** (2004) 181–221.
- [8] H. Alzer AND N. Batir, Monotonicity properties of the gamma function, *Appl. Math. Lett.*, **20** (7) (2007) 778–781.
- [9] H. Alzer, Inequalities for the harmonic numbers, *Math. Z.*, **267** (1–2) (2011) 367–384.
- [10] G. D. Anderson AND S. L. Qiu, A monotonicity property of the gamma function, *Proc. Amer. Math. Soc.*, **125** (11) (1997) 3355–3362.
- [11] N. Batir, Some new inequalities for gamma and polygamma functions, *J. Inequal. Pure Appl. Math.*, **6** (4) (2005) Art. 103. . (Available online at <http://jipam.vu.edu.au/article.php?sid=577>)
- [12] N. Batir, On some properties of digamma and polygamma functions, *J. Math. Anal. Appl.*, **328** (1) (2007) 452–465.
- [13] N. Batir, Inequalities for the gamma function, *Arch. Math.* **91** (2008) 554–563.
- [14] N. Batir, Inequalities for the Gamma Function, *RGMIA Res. Rep. Coll.*, **12** (1) (2009) Art. 9.
- [15] N. Batir, Sharp bounds for the psi function and harmonic numbers, *Math. Inequal. Appl.*, In print.
- [16] C. Berg, Integral representation of some functions related to the gamma function, *Mediterr. J. Math.*, **1** (4) (2004) 433–439.
- [17] Ch. -P. Chen, Complete monotonicity and logarithmically complete monotonicity properties for the gamma and psi functions, *J. Math. Anal. Appl.*, **336** (2007) 812–822.
- [18] Ch. -P. Chen, Monotonicity properties of functions related to the psi function, *Appl. Math. Comput.*, **217** (7) (2010) 2905–2911.
- [19] Ch. -P. Chen, Sharpness of Negoi’s inequality for the Euler-Mascheroni constant, *Bull. Math. Anal. Appl.*, **3** (1) (2011) 134–141.
- [20] W. E. Clark AND M. E. H. Ismail, Inequalities involving gamma and psi function, *Anal. Appl.*, **1** (129) (2003) 129–140.
- [21] D. W. DeTemple and S.-H. Wang, Half integer approximations for the partial sums of the harmonic series, *J. Math.*
- [22] D. W. DeTemple, A quicker convergence to Euler’s constant, *Amer. Math. Monthly*, **100** (5) (1993) 468–470.
- [23] A. Elbert AND A. Laforgia, On some properties of the gamma function, *Proc. Amer. Math. Soc.*, **128** (9) (2000) 2667–2673.
- [24] B. J. English AND G. Rousseau, Bounds for certain harmonic sums, *J. Math. Anal. Appl.*, **206** (1997) 428–441.
- [25] S. Koumandos, Monotonicity of some functions involving the gamma and psi functions, *Math. Comp.*, **77** (2008) 2261–2275.
- [26] M. Merkle, A bibliography of gamma function and related topics, Ver. 0.4, 2010, <http://milanmerkle.com/documents/Gammabib-v.0.4.pdf>
- [27] C. Mortici, New approximations of the gamma function in terms of the digamma function, *Applied Mathematics Letters*, **23** (1) (2010) 97–100.
- [28] C. Mortici, Very accurate estimates of the polygamma functions, *Asymptot. Anal.*, **68** (3) (2010) 125–134.
- [29] C. Mortici, Improved convergence towards generalized Euler-Mascheroni constant, *Appl. Math. Comput.*, **215** (9) (2010) 3443–3448.

- [30] C. Mortici, Optimizing the rate of convergence in some new classes of sequences convergent to Euler's constant, *Anal. Appl.*, **8** (1) (2010) 99–107.
- [31] C. Mortici, A quicker convergence toward the gamma constant with the logarithmic term involving the constant e , *Carpathian J. Math.*, **26** (1) (2010) 86–91.
- [32] C. Mortici, Fast convergences towards Euler-Mascheroni constant, *Computational & Applied Mathematics*, **29** (3) (2010) 479–491. *Anal. Appl.*, **160** (1991) 149–156.
- [33] C. Mortici, New sharp bounds for gamma and digamma functions, *Annals of the Alexandru Ioan Cuza University - Mathematics*, **57** (1) (2011) 57–60.
- [34] C. Mortici, Accurate estimates of the gamma function involving the psi function, *Numer. Funct. Anal. Optim.*, **32** (4) (2011) 469–476.
- [35] T. Negoï, A faster convergence to the constant of Euler, *Gaz. Mat. Seria A*, **15** (1997) 111–113. (in Romanian)
- [36] F. Qi, AND B. -N. Guo, Some properties of the psi and polygamma functions, *Hacet. J. Math. Stat.*, **39** (2) (2010) 219–231.
- [37] S. -L. Qiu, AND M. Vuorinen, Some properties of the gamma and psi functions, with applications, *Math. Comp.*, **74** (250) (2005) 723–742.
- [38] L. Tóth, Asupra problemei C: 608 (On problem C: 608), *Gaz. Mat. Seria B*, **94** (8) (1989) 277–279.
- [39] H. Vogt AND J. Voigt, A monotonicity property of the Γ -function, *J. Inequal. Pure Appl. Math.*, **3** (5) (2002) Art. 73. (Available online at <http://jipam.vu.edu.au/article.php?sid=225>)
- [40] R. M. Young, Euler's constant, *Math. Mag.*, **75** (422) (1991) 187–190.

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